TM-1269
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Fermilab

Minimizing the Emittance in Designing the Lattice of an Electron Storage Ring

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A. Formulation

For a synchrotron radiation facility to get high spectral brilliance it is desirable to have a small emittance of the electron beam in the storage ring. It is well known that the horizontal emittance (the predominant emittance) of an electron beam in a storage ring is given by

$$\epsilon_x \equiv \frac{\sigma_x^2}{\beta_x} = \frac{C_q}{J_x}\frac{\gamma^2}{\rho}\langle \mathcal{H}\rangle_{dipole}$$

where

$$C_q = 3.832\times 10^{-13}\,m, \quad \gamma = \frac{E}{mc^2}, \quad \sigma_x = \text{rms beam width}$$

$$J_x = \text{horizontal partition factor} \cong 1 + \frac{1}{\gamma_t^2}\frac{R}{\rho},$$

and

$$\langle \mathcal{H}\rangle_{dipole} = \frac{1}{2\pi\rho}\int_{dipole}(\gamma\eta^2 + 2\alpha\eta\eta' + \beta\eta'^2)\,ds$$

$$\left\{ \begin{array}{l} \alpha,\ \beta,\ \gamma = \text{Courant-Snyder parameters} \\ \eta,\ \eta' = \text{dispersion functions} \end{array} \right.$$

(Hopefully the double useage of β and γ will not cause any confusion.) It is not difficult to see that $\langle \mathcal{H} \rangle_{\text{dipole}} \propto \frac{\ell^3}{\rho^2}$ for each dipole with length ℓ or

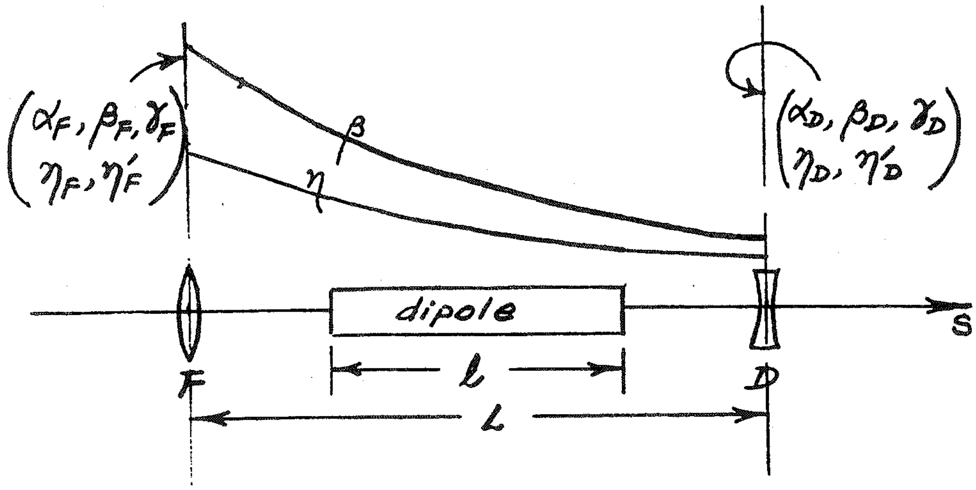
$$\epsilon_x = \frac{C_q}{J_x}\gamma^2\theta^3 F\ , \qquad F \equiv \frac{\rho^2}{\ell^3}\langle \mathcal{H} \rangle_{dipole}$$

where $\theta = \ell/\rho$ is the bending angle of each dipole assumed identical, and F is a numerical factor controlled by the lattice design.

For given energy (γ) ϵ_x is reduced by reducing θ or F. Reducing θ is very effective but requires using a large number of short dipoles. This has two undesirable side effects. First, the cost is higher for larger number of shorter magnets. Second, such an arrangement tends to take up more circumferential space and together with the long straight sections desired for the special insertions, will lead to a larger ring which will further increase the cost. Thus, the decision on θ is largely a social/political/economical one, and will not be further discussed here. Scientifically, however, one can ask the question that once θ is chosen what type of lattice will give the lowest F. This question we will now address.

B. FODO Lattice

The FODO lattice is far from optimal, but is used here to illustrate the computational procedure and to provide a point for comparison. As shown in the diagram we shall use the thin-lens approximate values for the orbit and the dispersion functions at the quadrupoles.



As a good approximation we shall neglect the very weak centrifugal focusing in the dipole. The functions in the first half of the dipole are then

$$\begin{cases} \beta = \beta_F - 2\alpha_F s + \gamma_F s^2 \\ \alpha = \alpha_F - \gamma_F s \\ \gamma = \gamma_F \end{cases} \qquad \begin{cases} \eta = \eta_F + \eta'_F s \\ \eta' = \eta'_F \end{cases}$$

This gives

$$\begin{aligned} \mathcal{H} &= \gamma\eta^2 + 2\alpha\eta\eta' + \beta\eta'^2 \\ &= \gamma_F(\eta_F + \eta'_F s)^2 + 2(\alpha_F - \gamma_F s)\eta'_F(\eta_F + \eta'_F s) + \eta'^2_F(\beta_F - 2\alpha_F s + \gamma_F s^2) \\ &= \gamma_F \eta_F^2 + 2\alpha_F \eta_F \eta'_F + \beta_F \eta'^2_F = \text{constant} \end{aligned}$$

For a FODO cell with phase advance μ we have

$$\alpha_F = \sqrt{\frac{1+x}{1-x}} \; ; \quad \beta_F = \frac{1}{x}\sqrt{\frac{1+x}{1-x}}\,L \, , \quad \gamma_F = \frac{2x}{\sqrt{(1+x)(1-x)}}\,\frac{1}{L}$$

and

$$\eta_F = \frac{1}{x}\left(\frac{1}{x}+\frac{1}{2}\right)\theta L \, , \qquad \eta_F' = -\left(\frac{1}{x}+\frac{1}{2}\right)\theta$$

where $x \equiv \sin(\mu/2)$. This gives

$$\langle \mathcal{H} \rangle = \left[\frac{2x}{\sqrt{(1+x)(1-x)}}\,\frac{1}{x^2}\left(\frac{1}{x}+\frac{1}{2}\right)^2 - 2\sqrt{\frac{1+x}{1-x}}\,\frac{1}{x}\left(\frac{1}{x}+\frac{1}{2}\right)^2 + \frac{1}{x}\sqrt{\frac{1+x}{1-x}}\left(\frac{1}{x}+\frac{1}{2}\right)^2\right]\theta^2 L$$

$$= \frac{1}{x}\sqrt{\frac{1-x}{1+x}}\left(\frac{1}{x}+\frac{1}{2}\right)^2\theta^2 L$$

or

$$F_{\frac{1}{2}} = \frac{1}{x}\sqrt{\frac{1-x}{1+x}}\left(\frac{1}{x}+\frac{1}{2}\right)^2\frac{L}{\ell} \qquad\qquad x = \sin\frac{\mu}{2}$$

For the second half of the dipole a similar procedure starting with variables with subscript D gives

$$F_{-\frac{1}{2}} = \frac{1}{x}\sqrt{\frac{1+x}{1-x}}\left(\frac{1}{x}-\frac{1}{2}\right)^2\frac{L}{\ell} \qquad\qquad x = \sin\frac{\mu}{2}$$

Averaging between these two halves we get finally

$$\boxed{F = \frac{1}{2x^3}\,\frac{4-3x^2}{\sqrt{1-x^2}}\,\frac{L}{\ell} = \frac{1}{2\sin\mu}\,\frac{5+3\cos\mu}{1-\cos\mu}\,\frac{L}{\ell}}$$

For commonly used values of the phase advance μ we have

$$\mu = 90° \qquad F = 2.50\,\frac{L}{\ell}$$

$$\mu = 72° \qquad F = 4.51\,\frac{L}{\ell}$$

$$\mu = 60° \qquad F = 7.51\,\frac{L}{\ell}.$$

$$\mu = 45° \qquad F = 17.19\,\frac{L}{\ell}.$$

We shall see below that even for $L/\ell = 1$ these values of F are rather large. This is expected because the FODO lattice is not optimized at all for this purpose.

C. Minimizing F

Again, we neglect the centrifugal focusing in a dipole and approximate both β and η as quadratic functions of s. It is clear that F is minimum when $\eta = 0$ and $\beta = \beta_0$ = minimum at the midpoint (s = 0) of the dipole. In this case

$$\begin{cases} \beta = \beta_0 + \dfrac{s^2}{\beta_0}, & \alpha = -\dfrac{s}{\beta_0}, \qquad \gamma = \dfrac{1}{\beta_0} \\ \eta = \dfrac{s^2}{2\rho}, & \eta' = \dfrac{s}{\rho} \end{cases}$$

and

$$\langle \mathcal{H} \rangle_{dipole} = \frac{1}{\ell}\int_{-\frac{\ell}{2}}^{\frac{\ell}{2}} \left(\frac{s^4}{4\beta_0\rho^2} - \frac{s^4}{\beta_0\rho^2} + \frac{s^4}{\beta_0\rho^2} + \beta_0\frac{s^2}{\rho^2} \right) ds$$

$$= \frac{1}{12}\,\frac{\ell^3}{\rho^2}\left(\frac{\beta_0}{\ell} + \frac{3}{80}\,\frac{\ell}{\beta_0} \right)$$

The expression in the parenthesis is a minimum when $\frac{\beta_0}{\ell} = \sqrt{\frac{3}{80}}$. This gives

$$\boxed{F_1^{min} = \frac{1}{8\sqrt{15}} = 0.0323 \quad \text{"Absolute minimum but useless"}}$$

This case is, however, rather unrealistic. Such a dispersion function will not produce a lattice which is appropriate for accommodating insertions. For this we want $\eta = \eta' = 0$ at one end of the dipole so as to produce a dispersion-free long straight section. We assume a totally general β-function which has a minimum of β_0 at $s = s_0 \neq 0$. In this case

$$\begin{cases} \beta = \beta_0 + \dfrac{(s-s_0)^2}{\beta_0^2}, & \alpha = -\dfrac{s-s_0}{\beta_0}, \quad \gamma = \dfrac{1}{\beta_0} \\ \eta = \dfrac{s^2}{2\rho}, & \eta' = \dfrac{s}{\rho} \end{cases}$$

and

$$\langle \mathcal{H} \rangle = \frac{1}{\ell}\int_0^\ell \left(\frac{1}{\beta_0}\frac{s^4}{4\rho^2} - \frac{s-s_0}{\beta_0}\frac{s^3}{\rho^2} + \frac{(s-s_0)^2}{\beta_0}\frac{s^2}{\rho^2} + \beta_0 \frac{s^2}{\rho^2} \right) ds$$

$$= \frac{1}{3}\frac{\ell^3}{\rho^2}\left[\frac{\beta_0}{\ell} + \frac{\ell}{\beta_0}\left(\frac{s_0^2}{\ell^2} - \frac{3}{4}\frac{s_0}{\ell} + \frac{3}{20} \right) \right]$$

The expression in the round parentheses has the minimum value of $\frac{3}{320}$ at $\frac{s_0}{\ell} = \frac{3}{8}$ and the whole expression is a minimum when

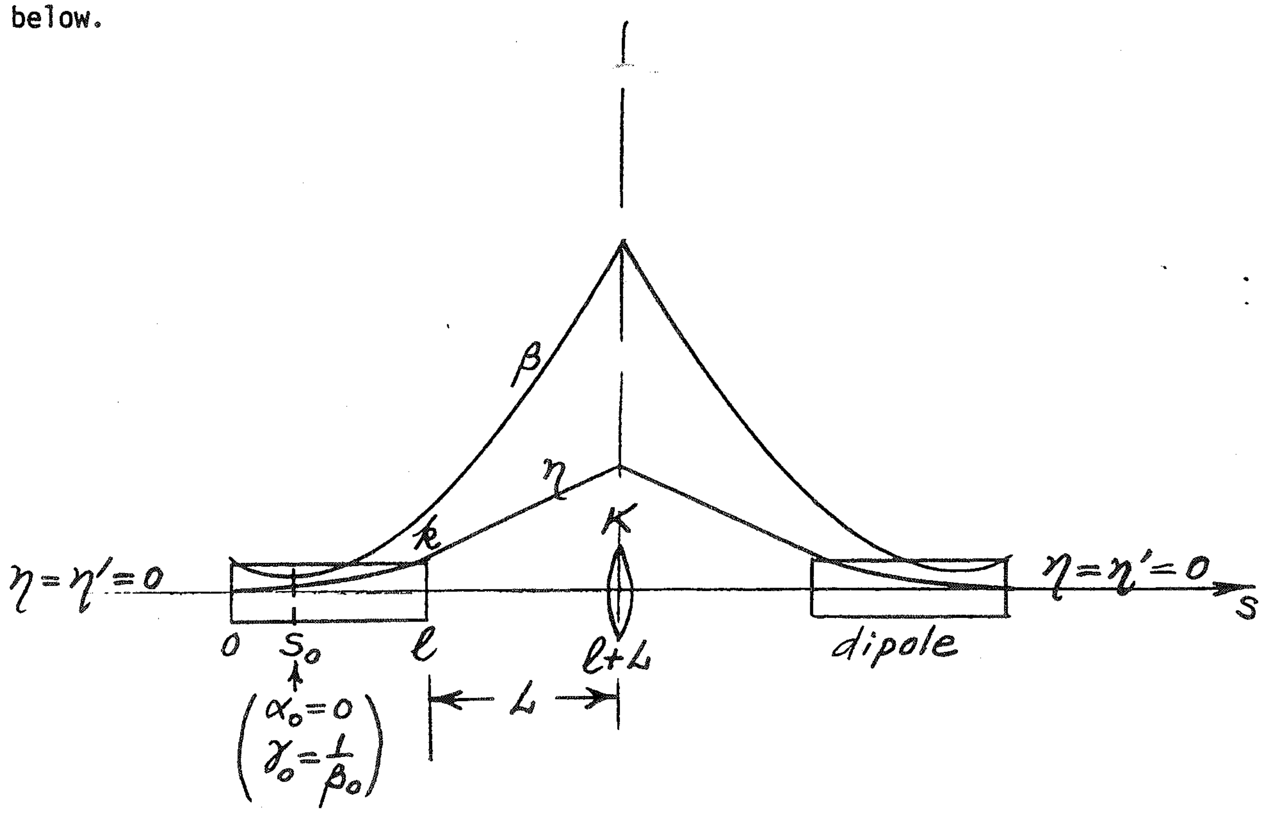
$$\frac{\beta_0}{\ell} = \left(\frac{s_0^2}{\ell^2} - \frac{3}{4}\frac{s_0}{\ell} + \frac{3}{20} \right)^{1/2} = \sqrt{\frac{3}{320}}$$

This gives

$$\boxed{F_2^{min} = \frac{1}{4\sqrt{15}} = 0.0645 \qquad \text{"useful but difficult"}}$$

Even, for this case, the optimal condition of $\frac{s_o}{\ell} = \frac{3}{8}$ is not too easy to obtain.

To make it even more realistic let us consider the symmetric achromatic bend connecting two dispersion-free straight sections shown below.



The focusing quadrupole in the middle has strength K and we have assumed an edge focusing k at the end of the dipole. The dispersion function to the left of K is given by

$$\eta = \frac{1}{2}\frac{\ell^2}{\rho} + L\left(\frac{\ell}{\rho} + \frac{1}{2}k\frac{\ell^2}{\rho}\right), \qquad \eta' = \frac{\ell}{\rho} + \frac{1}{2}k\frac{\ell^2}{\rho}$$

The transfer matrix from s_o to ℓ+L is

$$\begin{pmatrix} 1 & L \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ k & 1 \end{pmatrix}\begin{pmatrix} 1 & \ell - s_0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1+kL & L+(1+kL)(\ell-s_0) \\ k & 1+k(\ell-s_0) \end{pmatrix}$$

Hence the orbit functions to the left of K are

$$\begin{cases} \beta = (1+kL)^2\beta_0 + [L+(1+kL)(\ell-s_0)]^2 \dfrac{1}{\beta_0} \\ -\alpha = k(1+kL)\beta_0 + [1+k(\ell-s_0)][L+(1+kL)(\ell-s_0)]\dfrac{1}{\beta_0} \end{cases}$$

The matching conditions at K are

$$K\eta = -2\eta' \qquad \text{and} \qquad K\beta = 2\alpha$$

or, by eliminating K, $\beta\eta' = -\alpha\eta$. This gives

$$\left\{(1+kL)^2\beta_0^2 + [L+(1+kL)(\ell-s_0)]^2\right\}\left(1+\tfrac{1}{2}k\ell\right)$$
$$= \left\{k(1+kL)\beta_0^2 + [1+k(\ell-s_0)][L+(1+kL)(\ell-s_0)]\right\}\left(\tfrac{\ell}{2}+L+\tfrac{\ell}{2}kL\right)$$

or

$$\beta_0^2 = \left(\frac{L}{1+kL} + \ell - s_0\right)\left(s_0 - \frac{\ell}{2}\right)$$

For reasonable values of k this requires $\frac{s_0}{\ell} > \frac{1}{2}$ or

$$\boxed{F > F_3^{min} = \frac{1}{3\sqrt{10}} = 0.1054 \qquad \text{"useful and realistic"}}$$

It is important to note that this realistic case gives an F_3^{min} which is only about 1.6 times larger than the unrealistic minimum value F_2^{min}. All these F values are about two orders of magnitude smaller than those of the corresponding FODO cells.

D. The SRRC Lattice

For the TLS we have

γ = 1958 (1 GeV) $\quad$ J_x = 1.045

C_q = 3.832x10^{-13} m $\quad$ $\theta = \frac{\pi}{6}$

s_o = 0.78 m $\quad$ β_o = 0.32 m $\quad$ ℓ = 1.45 m

This gives

$$F = \frac{1}{3}\left[\frac{\beta_0}{\ell} + \frac{\ell}{\beta_0}\left(\frac{s_0^2}{\ell^2} - \frac{3}{4}\frac{s_0}{\ell} + \frac{3}{20}\right)\right] = 0.1278$$

and

$$\epsilon_x = \frac{3.832 \times 10^{-13}\,m}{1.045} \times 1958^2 \times \left(\frac{\pi}{6}\right)^3 \times 0.1278$$

$$= 2.58 \times 10^{-8}\,m$$

The point is that the F value of 0.1278 is larger than the theoretical minimum F_3^{min} = 0.1054 by only about 20%. It is, therefore safe to state that unless one increases the number of dipoles as was done for the LBL-ALS the design emittance of TLS is very nearly the best possible.

To summarize, the present TLS design gives an emittance which is

4 x "Theoretical minimum but useless".

2 x "Useful minimum but difficult".

1.2 x "Useful and realistic minimum".

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<u>Addendum to "Minimizing the Emittance in Designing the Lattice of an Electron Storage Ring - TM-1969"</u>

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The approximation used in TM-1269 of taking both the horizontal β and η in a dipole as quadratic functions of s is good for bending angles much smaller than unity. For the TLS the bending angle of $\pi/6$ is too large for this to be a quantitatively good approximation, although all qualitative conclusions remain valid. We derive here the exact formulas which apply to large bending angles.

Again we take s=o to be the end of the dipole where $\eta=\eta'=0$, and $s=s_0$ to be the location where β has the minimum value β_0 and where $\alpha_0 = 0$ and $\gamma_0 = \frac{1}{\beta_0}$. The tranfer matrix from s_0 to s is then

$$M = \begin{pmatrix} \cos\frac{s-s_0}{\rho} & \rho\sin\frac{s-s_0}{\rho} \\ -\frac{1}{\rho}\sin\frac{s-s_0}{\rho} & \cos\frac{s-s_0}{\rho} \end{pmatrix},$$

and the exact betatron functions are

$$\left\{\begin{aligned} \beta &= \beta_0\cos^2\frac{s-s_0}{\rho} + \frac{\rho^2}{\beta_0}\sin^2\frac{s-s_0}{\rho} \\ \alpha &= \left(\frac{\beta_0}{\rho} - \frac{\rho}{\beta_0}\right)\sin\frac{s-s_0}{\rho}\cos\frac{s-s_0}{\rho} \\ \gamma &= \frac{\beta_0}{\rho^2}\sin^2\frac{s-s_0}{\rho} + \frac{1}{\beta_0}\cos^2\frac{s-s_0}{\rho}. \end{aligned}\right.$$

The exact dispersion functions are

$$\begin{cases} \eta = \rho\left(1 - \cos\frac{s}{\rho}\right) \\ \eta' = \sin\frac{s}{\rho} . \end{cases} \tag{3}$$

Substituting these we get

$$\begin{aligned} \frac{1}{\rho}\mathcal{H} &\equiv \frac{1}{\rho}\left(\gamma\eta^2 + 2\alpha\eta\eta' + \beta\eta'^2\right) \\ &= \left[\frac{\rho}{\beta_0} + \left(\frac{\beta_0}{\rho} - \frac{\rho}{\beta_0}\right)\sin^2\frac{s-s_0}{\rho}\right]\left(1-\cos\frac{s}{\rho}\right)^2 \\ &+ 2\left(\frac{\beta_0}{\rho} - \frac{\rho}{\beta_0}\right)\sin\frac{s-s_0}{\rho}\cos\frac{s-s_0}{\rho}\sin\frac{s}{\rho}\left(1-\cos\frac{s}{\rho}\right) \\ &+ \left[\frac{\rho}{\beta_0} + \left(\frac{\beta_0}{\rho} - \frac{\rho}{\beta_0}\right)\cos^2\frac{s-s_0}{\rho}\right]\sin^2\frac{s}{\rho} . \end{aligned} \tag{4}$$

From this we obtain, after some calculation

$$\begin{aligned} \frac{1}{\rho}\langle\mathcal{H}\rangle &= \frac{1}{\rho}\frac{1}{\ell}\int_0^\ell \mathcal{H}\,ds = 2\frac{\rho}{\beta_0}\left(1 - \frac{\rho}{\ell}\sin\frac{\ell}{\rho}\right) \\ &+ \frac{\rho}{\ell}\left(\frac{\beta_0}{\rho} - \frac{\rho}{\beta_0}\right)\left[\frac{3}{4}\sin\frac{2s_0}{\rho} + \frac{\ell}{\rho}\left(1 - \frac{1}{2}\cos\frac{2s_0}{\rho}\right)\right. \\ &\left. - 2\sin\frac{s_0}{\rho}\cos\frac{\ell - s_0}{\rho} - \frac{1}{4}\sin 2\frac{\ell - s_0}{\rho}\right] \\ &\equiv (A+B)\frac{\beta_0}{\rho} + (A-B)\frac{\rho}{\beta_0} , \end{aligned} \tag{5}$$

where

$$\begin{cases} A \equiv 1 - \frac{\rho}{\ell}\sin\frac{\ell}{\rho} \\ B \equiv \frac{1}{4}\frac{\rho}{\ell}\Big[\left(\cos 2\frac{\ell}{\rho} - 4\cos\frac{\ell}{\rho} + 3\right)\sin 2\frac{s_0}{\rho} \\ \qquad - \left(\sin 2\frac{\ell}{\rho} - 4\sin\frac{\ell}{\rho} + 2\frac{\ell}{\rho}\right)\cos 2\frac{s_0}{\rho}\Big] . \end{cases}$$

To minimize $\frac{1}{\rho} < H >$ or the emittance

$$\epsilon = \frac{C_q}{J_x}\gamma^2\frac{1}{\rho}\langle \mathcal{H} \rangle \qquad \text{with} \qquad C_q = 3.832 \times 10^{-13} m,$$

we must have

$$\left(\frac{\beta_0}{\rho}\right)_{min\,\epsilon} = \sqrt{\frac{A-B}{A+B}}$$

which gives

$$\frac{1}{\rho}\langle \mathcal{H} \rangle_{min} = 2\sqrt{A^2 - B^2}$$

or

$$\epsilon_{min} = \frac{2C_q}{J_x}\gamma^2\sqrt{A^2-B^2} .$$

Thus with given ℓ and ρ one must adjust s_0 to minimize A^2-B^2. For this we need

$$\frac{d}{ds_o}\left(A^2-B^2\right) = -2B\frac{dB}{ds_o} = 0 \ . \tag{11}$$

It is easy to show that B=o does not lead to physically reasonable solutions, and $\frac{dB}{ds_o}$ = o gives

$$\tan 2\left(\frac{s_o}{\rho}\right)_{\min\epsilon} = -\frac{\cos 2\frac{\ell}{\rho} - 4\cos\frac{\ell}{\rho} + 3}{\sin 2\frac{\ell}{\rho} - 4\sin\frac{\ell}{\rho} + 2\frac{\ell}{\rho}} \ . \tag{12}$$

For small $\frac{\ell}{\rho}$ and $\frac{s_o}{\rho}$ this gives

$$\tan 2\left(\frac{s_o}{\rho}\right)_{\min\epsilon} \cong 2\left(\frac{s_o}{\rho}\right)_{\min\epsilon} \cong \frac{3}{4}\frac{\ell}{\rho} \tag{13}$$

or

$$\left(s_o\right)_{\min\epsilon} \cong \frac{3}{8}\ell \ . \tag{14}$$

agreeing with the approximate result given in TM-1269. For $\frac{\ell}{\rho} = \frac{\pi}{6}$ and ρ = 2.769 m as in the TLS Eqs. (12), (8) and (10) give

$$\left\{\begin{array}{l} \left(s_o\right)_{\min\epsilon} = 0.543\ m = 0.3746\,\ell \\ \left(\beta_c\right)_{\min\epsilon} = 0.0508\ m \\ \epsilon_{\min} = 1.34\times10^{-8}\ m \quad (\text{ assuming } J_x \cong 1) . \end{array}\right. \tag{15}$$

These values of s_o and β_o will, however, lead to unrealistically high values of β on either side of the dipole. In the TLS design we compromised by going to higher values of β_o and s_o, and settled for an emittance about twice ε_{min}.

Frequently one has to calculate s_o and β_o from the values β_1 (at s=o) and β_2 (at s=ℓ). For this we have

$$\begin{cases} \dfrac{\beta_1}{\rho} = \dfrac{\beta_0}{\rho}\cos^2\dfrac{s_0}{\rho} + \dfrac{\rho}{\beta_0}\sin^2\dfrac{s_0}{\rho} \\[2ex] \dfrac{\beta_2}{\rho} = \dfrac{\beta_0}{\rho}\cos^2\dfrac{\ell-s_0}{\rho} + \dfrac{\rho}{\beta_0}\sin^2\dfrac{\ell-s_0}{\rho} \end{cases}$$

or, after solving for β_o and s_o

$$\begin{cases} \dfrac{\beta_0}{\rho} = C - \sqrt{C^2-1} \\[2ex] \qquad \text{where} \quad C = \dfrac{1}{\sin^2\frac{\ell}{\rho}}\left[\dfrac{\beta_1+\beta_2}{2\rho} + \cos\dfrac{\ell}{\rho}\sqrt{\dfrac{\beta_1\beta_2}{\rho^2} - \sin^2\dfrac{\ell}{\rho}}\right] \\[2ex] \sin\dfrac{2s_0-\ell}{\rho} = \dfrac{\beta_1-\beta_2}{\rho\sin\frac{\ell}{\rho}}\,\dfrac{1}{\frac{\rho}{\beta_0}-\frac{\beta_0}{\rho}} \end{cases}$$

For the TLS design values we have

$$\beta_1 = 2.120 \text{ m}, \qquad \beta_2 = 1.711 \text{ m}.$$

This gives through Eqs. (17)

$$s_0 = 0.7727 \text{ m}, \qquad \beta_0 = 0.3187 \text{ m}$$

and through Eqs. (5), (6), and (7)

$$\frac{1}{\rho}\langle \mathcal{H} \rangle = 0.01704 \quad \text{and} \quad \epsilon = 2.591\times 10^{-8}\,m. \tag{20}$$

This emittance agrees with that given by the computer program.